Supplementary appendix to "Unions and market integration in contests"

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Uniqueness of the symmetric equilibrium

We have reasonably assumed that at least one symmetric equilibrium exists. What remains to be shown is that such an equilibrium is unique. A sufficient condition for uniqueness is that the marginal payoff vector, $M(\overrightarrow{w}) = [\frac{\partial E[U_1(\overrightarrow{w})]}{\partial w_1}, \frac{\partial E[U_2(\overrightarrow{w})]}{\partial w_2}, ..., \frac{\partial E[U_n(\overrightarrow{w})]}{\partial w_n}]$, where $\overrightarrow{w}$ is some arbitrary vector, $[w_1, w_2, ..., w_n]$ is strictly monotonically decreasing. I.e. for another vector $\overrightarrow{w}^* = [w_1^*, w_2^*, ..., w_n^*] \neq \overrightarrow{w}$:

$$[M(\overrightarrow{w}) - M(\overrightarrow{w}^*)] \cdot [\overrightarrow{w} - \overrightarrow{w}^*] =$$
$$\sum_{j=1}^{n} [\frac{\partial E[U_j(\overrightarrow{w})]}{\partial w_j} - \frac{\partial E[U_j(\overrightarrow{w}^*)]}{\partial w_j}][w_j - w_j^*] < 0 \tag{1}$$

In any symmetric equilibrium, $\overrightarrow{w}^*$, wages are equal across unions. Furthermore, $M(\overrightarrow{w}^*) = \overrightarrow{0}$. Thus the above condition reduces to

$$n\frac{\partial E[U_i(\overrightarrow{w})]}{\partial w_i}[w_i - w^*] < 0 \tag{2}$$

Certainly these conditions for uniqueness are met if $E[U_i(\overrightarrow{w})]$ can be shown to be strictly concave in union i's wages. Next, we show this to be the case in our setting. We have:

$$\frac{\partial^2 E[U_i]}{\partial w_i^2} = 2\frac{\partial}{\partial w_i}P + w\frac{\partial^2}{\partial w_i^2}P \tag{3}$$

Furthermore:

$$\frac{\partial}{\partial w_i}P = \frac{S_n}{(V-w_i)^2}[\frac{S_n}{(n-1)(V-w_i)} - 1] \tag{4}$$

This gives us:

$$\begin{aligned}
\frac{\partial^2}{\partial w_i^2}P &= \frac{(V-w_i)\frac{\partial S_n}{\partial w_i} + 2S_n}{(V-w_i)^3}[\frac{S_n}{(n-1)(V-w_i)} - 1] \\
&+ \frac{S_n}{(n-1)(V-w_i)^2}\frac{(V-w_i)\frac{\partial S_n}{\partial w_i} + S_n}{(V-w_i)^2} \\
&= \frac{1}{(V-w_i)^3}\{[\frac{2S_n}{(n-1)} - (V-w_i)]\frac{\partial S_n}{\partial w_i} \\
&+ \frac{3(S_n)^2}{(n-1)(V-w_i)} - 2S_n\}
\end{aligned} \tag{5}$$

Inserting these expressions into (3), we obtain:

$$\begin{aligned}
\frac{\partial^2 E[U_i]}{\partial w_i^2} &= 2\frac{S_n}{(V-w_i)^2}(\frac{S_n}{(n-1)(V-w_i)} - 1) \\
&+ \frac{w_i}{(V-w_i)^3}((\frac{2S_n}{(n-1)} - (V-w_i)) \\
&(-S_n^2\frac{1}{(n-1)(V-w_i)^2}) + \frac{3(S_n)^2}{(n-1)(V-w_i)} - 2S_n)
\end{aligned} \tag{6}$$

Some simplifications yield:

$$\frac{\partial^2 E[U_i]}{\partial w_i^2} = \frac{2S_n}{(V-w_i)^2}[\frac{S_n}{(V-w_i)(n-1)} - 1 - \frac{w_i}{(V-w_i)}(\frac{S_n}{(n-1)(V-w_i)} - 1)^2] \tag{7}$$

Now, we already have an expression for total effort given wages: $S_n = \frac{n-1}{\sum_{j=1}^{n}\frac{1}{V-w_j}}$. From this it is it is easily shown that $\frac{S_n}{(V-w_i)(n-1)} - 1 < 0:$

$$S_n = \frac{n-1}{\sum_{j=1}^{n}\frac{1}{V-w_j}} \Leftrightarrow \tag{8}$$

$$\frac{S_n}{(V-w_i)(n-1)} = \frac{1}{(V-w_i)\sum_{j=1}^{n}\frac{1}{V-w_j}} = \frac{1}{1+\sum_{j\neq i}\frac{V-w_i}{V-w_j}} < 1 \tag{9}$$

This proves that for all $w_i < V$, expected utility is concave in own wages, and thus the symmetric equilibrium is unique.